

Re-Understanding the Essence of Calculus Operations and Analysis of Its Number Field Foundations

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Abstract: This study summarizes the series of confusions that arise when studying calculus theory within the traditional limit-based frameworks. The study analyzes several questionable aspects of constructing calculus systems based on limits from eight dimensions: rules of the division operation, objective laws of partition, differential concepts, logical deduction, semantic connotations, learning and practice, the transition from quantitative to qualitative change, and numerical field operations. It argues that the limit is merely an outward manifestation of calculus rather than its essence, introduces the concepts of explicit and implicit real fields, explains the internal operation mechanism of differentiation and integration, and contends that calculus is an inherent natural property of real numbers, and is a mathematical discovery rather than a human invention, in order to construct a cognitive system of calculus that can be connected with elementary mathematical operation rules, and provide a new perspective for understanding the underlying logic of calculus.

Keywords: Calculus; Limit; Implicit real field; Differential; Integral

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1. Introduction

From the moment the author first encountered calculus, the author hoped to thoroughly understand and comprehend it. But things did not go as planned. The deeper one thinks, the more confused one becomes, and many problems are still difficult to figure out. The traditional calculus theory is based on limits, and this theoretical framework is widely adopted in mainstream mathematical analysis textbooks in China^[1-2]. Since the limit exists as $\Delta x \rightarrow 0$, this theory then takes (or defines) $\Delta x = 0$ and thereby derives the quotient $\frac{\Delta y}{\Delta x}$, i.e., the derivative. However, this derivation logic still deserves further scrutiny.

2. Eight problems with constructing calculus on limits

- (1) The divisor Δx cannot be 0, which is basic knowledge established in elementary mathematics. Although Δx is a variable, when it is involved in the multiplication and division of functions, it must satisfy the operational constraints of multiplication and division. The derivative is a result derived from the rules governing division, rather than a result obtained through simple manipulation of variables. Calculus belongs to mathematical operational rules and should not be separated from basic arithmetic rules. There is a mathematical-philosophical point here worth exploring.
- (2) Mathematics is consistent with the laws governing objective things, and there is a clear boundary between existence and non-existence. The two cannot be confused with each other. Just as matter can be divided into molecules, and molecules can be further divided into atoms, the behavior of division can only hold within the corresponding category; one cannot partition an existing quantity into a non-existent one; the quantity of “existence” as “non-existence.” Therefore, the approach of directly taking $\Delta x = 0$ in the limit conflicts with objective partition laws.
- (3) If $\Delta x = 0$, the differential loses its meaning as a minute unit of partition. 0 is only a special individual case of the differential and cannot represent the general form of the differential. Substituting special cases for general concepts has logical flaws.
- (4) In the process of finding the quotient function, i.e., the derivative, the initial definition requires that Δx tends to 0 and is not equal to 0, which is a necessary prerequisite for division operations to be carried out. However, in actual calculation and processing, Δx is set to 0, and the two different states of $\Delta x \neq 0$ and $\Delta x = 0$ are merged into the same set of equations, which is internally inconsistent. In fact, Δx cannot be equal to 0. To clarify this issue, it is necessary to review the development history of calculus: which came first, calculus or limit theory? What was the original calculus calculation idea of Newton and Leibniz? Research in the history of mathematics shows that calculus operational rules were born before the rigorous theory of limits^[3]. In the operation, only the numerator can take 0; that is, the increment of the constant function $y = c$ is 0, and its differential change is equal to 0.
- (5) From a semantic perspective, for the operation $\frac{dy}{dx}$, if $dx=0$, the expression becomes meaningless. Although the formal derivation of limit theory is rigorous, the final processing effect is equivalent to setting $\Delta x = 0$. This raises the questions: What is the object of differential operation? What is the object of integration^[4]?
- (6) Calculus based on limits is difficult to learn, compute with, understand and remember, which can easily transform simple calculation problems into complex speculative deductions, causing confusion in conceptual understanding. Xu Lizhi once pointed out that the concept of limits is one of the main difficulties faced by beginners in calculus^[5].
- (7) Calculus constructed with limits has severed the continuity of mathematical operational rules and failed to fully reveal the inherent essence and evolution process of operations from quantitative change to qualitative change. In fact, the complete path for a continuous function to approach 0 is that the function first enters the “implicit real field” and then reaches point 0, which is the actual process by which a continuous function changes.
- (8) Constructing calculus with limits breaks the continuity between mathematical concepts and operational rules, and does not fully match traditional mathematical concepts, operational rules, and practical applications, leading to a tendency to complicate simple problems. In sum, the limit is merely an outward

manifestation of calculus, not its essence. Strictly speaking, the limit is a product of mathematical imagination rather than an operational rule. Limit-based calculus presents only the result for without exhibiting the complete computational process of “approaching 0”; that process is supplied by definition and stipulation, which leaves a degree of vagueness. The framework thus relies on artificial definitions to make up for gaps in the operational rules and to reach results that ordinary operations cannot, giving it a strongly subjective, constructed character. For uniformly varying quantities, calculations can be directly carried out; for non-uniform changes, especially for quantities with drastic changes, it is necessary to break them down into differentials in order to obtain accurate results. Calculus operations still use basic elementary rules such as addition, subtraction, multiplication, division, powers, and roots, essentially completing differential division and multiplication of continuous functions. From this, it can be concluded that differentiation is a division operation of functions in the implicit real field, and integration is a multiplication operation of functions in the implicit real field, in order to obtain accurate differential quotient and differential sum. The ordinary explicit real field cannot perform calculus operations, and calculus operations can only occur in the implicit real field. Division and the quotient per unit fit human intuition more naturally; division and its quotient were among the earliest operation–result pairs humans established. It is worth further exploring why the specialized concept of “derivative” is introduced to describe calculus operations.

3. Problems and essence of integral theory from the limit perspective

From a geometric perspective, the limit-based definition of the integral generally decomposes a curvilinear trapezoid into several small area elements and then sums them up to explain the area S of the curvilinear trapezoid $s = \lim_{\Delta x \rightarrow 0} \sum_{i=1}^n f(x_i) \Delta x_i$. However, this geometric interpretation is open to question. For a curvilinear trapezoid, two other partition methods can be used:

$$s = \lim_{\Delta y \rightarrow 0} \sum_{i=1}^n x_i \Delta y_i \text{ or } s = \lim_{\Delta f(x_i) \rightarrow 0} \sum_{i=1}^n x_i \Delta f(x_i)$$

According to the logic of limit theory, the above three types of partition and summation should yield the same result. However, in practical integration operations, in general:

$$\lim_{\Delta x \rightarrow 0} \sum_{i=1}^n f(x_i) \Delta x_i \neq \lim_{\Delta f(x_i) \rightarrow 0} \sum_{i=1}^n x_i \Delta f(x_i)$$

The difference between the two summation results is dxy , or, in other words, the difference is one $dxy(x)$, that is

$$\int y dx = - \int x dy + \int dxy$$

$$\int f(x) dx = - \int x dy(x) + \int dxy(x)$$

From this, it can be seen that integration cannot be obtained solely through limit deduction, and integration comes from objective operational facts. The integral is the sum of ordered differentials over a continuous domain, that is, the sum of ordered differentials $df(x)=f(x)dx$ within the continuous domain. Objectively speaking, there exists a corresponding antiderivative for the integrand function $f(x)$. The function of the antiderivative is to identify the differential order $f(x)$ of the integrand function $df(x)$. Starting

from the initial point $c = f(x_0)$, reassemble the differential units $f(x)dx = df(x)$ in differential order to complete the integration operation. This is not entirely equivalent to the textbook notion of “the family of functions $f(x) + c$ ”; the family is merely an outward mathematical representation, not the essence of integration. The antiderivative allows for a finite number of points of discontinuity within the domain, which do not change the integration calculation result. Integration is the inverse operation of differentiation, used to recover a function from its derivative. Differentiation and integration are fundamental dual operations in calculus, each defined in relation to the other, and together they provide a complete system for calculus computations.

4. The essence of calculus: The inherent properties of real numbers

With the development of times and scientific progress, researchers cannot be limited to the limit perspective to understand and apply calculus. Researchers need to conduct in-depth analysis within real numbers, understand calculus based on the inherent properties of real numbers, establish a logically clear mathematical thinking and calculation system, enhance the historical coherence of mathematical theory, and achieve the transition of mathematical thinking from quantitative to qualitative change.

How should researchers respond to the many questions raised above? Researchers need to look past the phenomena to the essence, break free from the limitations of thinking caused by limits, and explore the internal process of Δx approaching 0.

The New Theory of Calculus reveals the hidden “natural operational rules across number fields” within mathematics.

The establishment of calculus operations is rooted in the natural properties of real numbers themselves; calculus is a reflection of the inherent properties of real numbers at the mathematical level, and is not a rule that is artificially imposed. If real numbers did not possess this property, calculus theory could not be established. The implicit real field proposed in this article is different from the hyperreal field system established by Robinson in non-standard analysis ^[6].

5. Conclusion

Calculus strictly follows the dialectical law of quantitative change to qualitative change. From this perspective, calculus is a great mathematical discovery in human history, rather than a man-made mathematical invention. Newton and Leibniz first extended real number operations from the explicit real field to the implicit real field, achieving a qualitative leap in human cognition of number fields, establishing a calculus mode of thinking, gradually improving calculus calculation rules, and realizing unified operations across the explicit and implicit real fields as a unity of opposites. This is the essence of calculus. The limit theory remains at the cognitive level of the explicit real field, obscuring the important discovery of the implicit real field. It relies solely on subjective inference to complete the theoretical closure and fails to achieve a qualitative change at the cognitive level. Learning calculus is not just about mastering computational skills, but more importantly, establishing calculus awareness and thinking, conducting mathematical thinking within the unified framework of the explicit real field and the implicit real field, and achieving intellectual growth. Therefore, cultivating awareness and thinking of calculus is an important part of calculus learning.

The New Theory of Calculus fully explains this cognitive process, connects calculus with the four arithmetic operations of primary-school mathematics, and reflects the continuity and unity of mathematical operation rules.

Disclosure statement

The author declares no conflict of interest.

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